an RR will be obtained, while in steady flows a SMR, CMR, or DMR is expected in that region.

III. Conclusion

The present work on the reflection of oblique shock-waves in steady flows compliments the previous work on the pseudo-steady oblique shock-wave reflection.[2,3] Similarly, it establishes the domains of different types of reflection and their transition boundaries, and reveals the significance of real gas effect on shifting the boundary lines between the various domains. The domains and their boundaries are established for both perfect and imperfect monatomic (argon) and diatomic (nitrogen) gases. The present work for steady flows, as well as that already reported for pseudosteady flows, has brought new insight and order into these complex problems.

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Response of Duffing Oscillator to One Half-Cycle Sine Pulse

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Nomenclature

- A = amplitude
- f = transformation function
- F = P/m
- $J_\lambda(n\pi)$ = Bessel function of the first kind
- K = linear spring stiffness
- m = mass
- n = constant
- p = $\sqrt{K/m}$ = natural frequency
- P = magnitude of force
- t = time
- T = natural period
- X_S = P/K = static displacement
- x,y = displacements
- ϵ = μ/m = nonlinear parameter
- $\bar{\epsilon}$ = $\epsilon x_s^2/\omega^2$
- $\Gamma(\ )$ = gamma function
- λ = ultraspherical polynomial index
- μ = nonlinear spring stiffness
- ω = forcing frequency
- τ = pulse period
- θ = phase
- $(\dot{\ })$ = differentiation with respect to time

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Index categories: Aeroacoustics; Analytical and Numerical Methods; Vibration.

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Introduction

THE method of ultraspherical polynomial approximation has been extensively used to solve many nonlinear problems.[1-3] Here this method is utilized to obtain the response of the Duffing oscillator subjected to one half-cycle sine pulse. The corresponding linear problem is solved by Jacobsen.[4] It is shown that the method gives accurate results for nonresonant solutions, whereas it fails to give even qualitatively accurate solutions for resonant solutions. An example is solved and time-displacement solutions are obtained for many values of the ratio of pulse period to natural period. First peak and time taken for first peak are plotted vs τ/T.

Method

Consider the Duffing equation

$$m\ddot{x}+Kx+\mu x^3=P\sin\omega t \quad \text{for } t<\tau \tag{1}$$

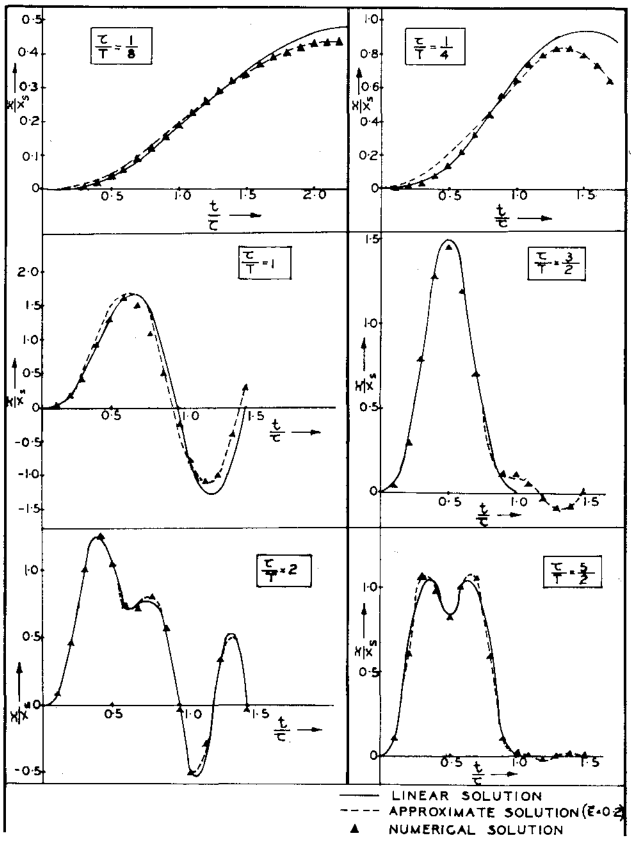

Fig. 1 Time-displacement curves for nonresonance solutions for various values of τ/T.

and

$$m\ddot{x}+Kx+\mu x^3=0 \quad \text{for } t>\tau \tag{2}$$

subject to initial conditions

$$x=0, \quad \dot{x}=0 \quad \text{at } t=0 \tag{3}$$

Dividing Eq. (1) throughout by m results in

$$\ddot{x}+p^2x+\epsilon x^3=F\sin\omega t \tag{4}$$

Let

$$x=y+f \tag{5}$$

be the solution of Eq. (4), where f is the solution of

$$\ddot{f}+p^2f=F\sin\omega t$$

that is,

$$f=F\sin\omega t/p^2-\omega^2 \tag{6}$$

Substitution of Eq. (6) in Eq. (4) results in

$$\ddot{y}+p^2y+\epsilon(y+f)^3=0 \tag{7}$$

and initial conditions (3) become

$$y=-f|_{t=0}, \quad \dot{y}=-\dot{f}|_{t=0} \tag{8}$$

Assume

$$y=A\cos\psi \tag{9}$$

where $\psi=pt+\theta$ as the solution of Eq. (7). Now, following Anderson[5] results in

$$\dot{A}=\frac{\epsilon}{p}\left[A^3\left\{\frac{\sin 4\psi}{8}+\frac{\sin 2\psi}{4}\right\}+\frac{3A^2F\sin\omega t}{p^2-\omega^2}\left(\frac{\sin 3\psi}{2}\right)+\frac{3AF^2\sin^2\omega t}{(p^2-\omega^2)^2}\left(\frac{\sin 2\psi}{2}\right)+\frac{F^3\sin^3\omega t}{(p^2-\omega^2)^3}\sin\psi\right]$$

$$\dot{\theta}=\frac{\epsilon}{pA}\left[A^3\left(\frac{3}{8}+\frac{\cos 2\psi}{2}+\frac{\cos 4\psi}{2}\right)+\frac{3A^2F\sin\omega t}{p^2-\omega^2}\left(\frac{\cos 3\psi}{4}+\frac{3}{4}\cos\psi\right)+\frac{3AF^2\sin^2\omega t}{(p^2-\omega^2)^2}\left(\frac{1+\cos 2\psi}{2}\right)+\frac{F^3\sin^3\omega t}{(p^2-\omega^2)^3}\cos\psi\right]$$

Expanding the right-hand side in ultraspherical polynomials and retaining only the first term leads to

$$\dot{A}=0$$

$$\dot{\theta}=\frac{\epsilon}{pA}\left[A^3\left(\frac{3}{8}+\frac{C_2}{2}+\frac{C_4}{8}\right)+\frac{3A^2F\sin\omega t}{p^2-\omega^2}\left(\frac{C_3}{4}+\frac{3C_1}{4}\right)+\frac{3AF^2\sin^2\omega t}{(p^2-\omega^2)^2}\left(\frac{1+C_2}{2}\right)+\frac{F^3\sin^3\omega t}{(p^2-\omega^2)^3}C_1\right]$$

where

$$C_n=\frac{\Gamma(\lambda+1)J_\lambda(n\pi)}{(n\pi/2)^\lambda}$$

Integrating the above equations, applying initial conditions (8), and using Eqs. (5) and (9) leads to

$$\theta=\bar{\epsilon}\left[\frac{\omega^3/p^3}{[1-(\omega^2/p^2)]^2}\left\{\frac{3}{8}+\frac{C_2}{2}+\frac{C_4}{8}+\frac{3(1+C_2)}{4(\omega^2/p^2)}\right\}\frac{t}{\tau}\pi-\frac{3\omega/p}{[1-(\omega^2/p^2)]^2}\left(\frac{1+C_2}{8}\right)\sin 2\pi\left(\frac{t}{\tau}\right)\right.$$
$$-\left\{\frac{3(\omega^2/p^2)}{[1-(\omega^2/p^2)]^2}\left(\frac{C_3+3C_1}{4}\right)+\frac{3C_1}{4[1-(\omega^2/p^2)]^2}\right\}\times\left\{\cos\left(\frac{t}{\tau}\right)\pi-1\right\}+\frac{C_1}{4[1-(\omega^2/p^2)]^2}$$
$$\left.\times\left\{\frac{\cos 3\pi(t/\tau)-1}{3}\right\}\right]+\frac{\pi}{2} \tag{10}$$

$$\frac{x}{X_S}=\left[\sin\pi\left(\frac{t}{\tau}\right)+\frac{\omega}{p}\cos\psi\right]\Big/\left(1-\frac{\omega^2}{p^2}\right) \tag{11}$$

where

$$\psi=pt+\theta \tag{12}$$

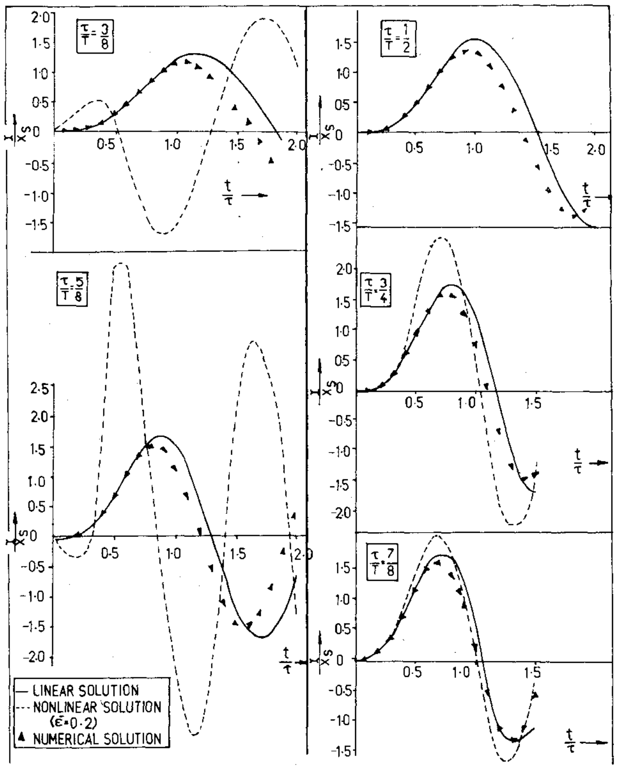

Fig. 2 Time-displacement curves for resonance solutions for various values of τ/T.

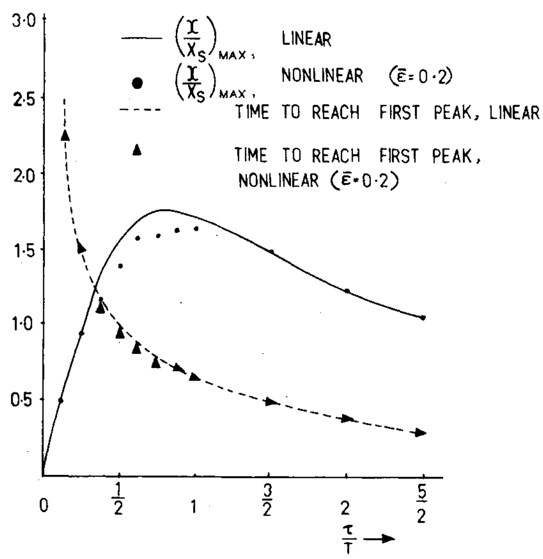

Fig. 3 Plot of first peak and time taken for first peak vs τ/T.

Equation (11), together with Eqs. (10) and (12), gives the solution for Eq. (1).

An example with nonlinearity parameter $\bar\epsilon = 0.2$ is worked out for various values of τ/T. In Fig. 1, time-displacement relations are plotted for nonresonance solutions for various values of τ/T, with ultraspherical polynomial index $\lambda = 1/2$. It is seen that the method gives results which compare well with numerical solution, obtained by using a Runge-Kutta fourth-order algorithm on the PACER 600 computer. The linear solution is also plotted in Fig. 1.

Limitation of the Approximation Method

The approximation method fails to give accurate results for resonance solutions. For the values of $\tau/T = 3/8$, $1/2$, and $5/8$, it fails to give even qualitative results; for $\tau/T = 3/4$ and $7/8$, it gives quantitatively wrong results. In Fig. 2, time-displacement relations are plotted for resonance solutions for various values of τ/T.

In Fig. 3, first peak and time taken to reach first peak are plotted vs τ/T. For nonresonance solutions, results are taken from approximate solution and for resonance solutions, it is taken from numerical solution. It is seen that maximax response occurs at about $\tau/T = 0.7$.

Conclusions

1) The method of ultraspherical polynomial approximation gives good results for nonresonance solutions.

2) It completely fails at and around resonance, that is, at $\tau/T = 0.5$.

3) The maximax response occurs at about $\tau/T = 0.7$.

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Low-Frequency and Small Perturbation Equation for Transonic Flow Past Wings

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Introduction

MOST methods for transonic flow calculations for arbitrary lifting wings and moderately complicated wing-fuselage combinations use transonic small perturbation theory, but differ in model equation.[1] Their shock relations have been investigated in Ref. 1 and compared with that of full potential equations and the Rankine-Hugoniot relation. It is apparent from the comparison that the equation used in NLR is the best of the methods for the calculation of flows with shocks past finite wings. Couston and Angelini[2] presented an original approach to derive a two-dimensional, low-frequency small perturbation equation and its corresponding boundary condition which avoids some of the arbitrariness of the equation used in Ref. 3 by using the concept of weak solution. Recently, Schmidt[4] formulated a self-consistent transonic small disturbance equation and boundary condition along the same line as that in Ref. 1. Making the approximations at the functional level and using the concept of weak solution as was done in Ref. 2 for a two-dimensional case, the present Note gives the small perturbation equations and the boundary conditions for moderate aspect ratio finite wings. The corresponding modifications for large aspect ratio and low aspect ratio slender wings can be carried out from the order of magnitude of different terms in different cases.

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Derivation of Small Perturbation Equations

Consider a three-dimensional unsteady potential flow past an arbitrary wing in a Cartesian coordinate system X, Y, and Z. The X axis coincides with the freestream direction Y along the spanwise direction, and Z is perpendicular to the X-Y plane to form a right-handed system with the origin at the leading edge of the root chord. The equation for the conservation of mass in terms of the velocity potential Φ and density $\bar\rho$ can be written as

$$\frac{\partial \bar\rho}{\partial \bar t} + \text{div}\,(\bar\rho\,\overrightarrow{\text{grad}}\Phi) = 0 \tag{1}$$

The Bernouilli equation gives

$$\frac{\partial\Phi}{\partial\bar t} + \frac{1}{2}[\overrightarrow{\text{grad}}^2\Phi - U_\infty^2] + \frac{(\bar\rho/\rho_\infty)^{\gamma-1} - 1}{\gamma - 1}c_\infty^2 = 0 \tag{2}$$

where γ is the ratio of the specific heats, c_∞ is the speed of sound at infinity, U_∞ is the freestream velocity, and $\bar t$ is the time.

The desired solution should have the property that Φ and its first derivatives are piecewise continuous, satisfying the mass conservation law [see Eq. (1)] at points where the flow is smooth together with a jump condition across a discontinuity. That is to say, Φ should be the weak solution of Eq. (1). Interest is restricted to the solution of periodic flow motion with small amplitude. For any periodic test function ψ, the weak equation corresponding to Eq. (1) is written

$$\iiiint\left(\bar\rho\frac{\partial\psi}{\partial\bar t} + \bar\rho\,\overrightarrow{\text{grad}}\Phi\cdot\overrightarrow{\text{grad}}\psi\right)\text{d}X\text{d}Y\text{d}Z\text{d}\bar t = 0 \tag{3}$$

The solution of Eq. (3) will satisfy Eq. (1) in the weak sense and the jump conditions across any discontinuity.[5] Consider the functional

$$\mathcal{L}(\Phi) = \iiiint \frac{(\bar\rho/\rho_\infty)^{\gamma} - 1}{\gamma M_\infty^2}\text{d}X\text{d}Y\text{d}Z\text{d}\bar t \tag{4}$$

where, from Eq. (2),

$$(\bar\rho/\rho_\infty)^{\gamma} = \left\{1 - \frac{\gamma-1}{c_\infty^2}\left[\frac{\partial\Phi}{\partial\bar t} + \frac{1}{2}(\overrightarrow{\text{grad}}^2\Phi - U_\infty^2)\right]\right\}^{\frac{\gamma}{\gamma-1}} \tag{5}$$

It can be shown that the Gateaux-difference

$$\delta\mathcal{L}(\Phi) = \lim_{\bar\lambda\to 0}\left[\frac{\mathcal{L}(\Phi+\bar\lambda\psi) - \mathcal{L}(\Phi-\bar\lambda\psi)}{2\bar\lambda}\right] \tag{6}$$